Factorizable quantum channels, non-closure of quantum correlations and the Connes Embedding Problem

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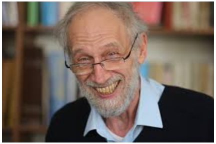


The Connes Embedding Problem (CEP)(Annals of Math.'76): Does every separable finite von Neumann alg M admit an embedding into

$$\mathcal{R}^\omega = \ell^\infty(\mathcal{R})/\{(T_n) : \lim_\omega \|T_n\|_2 = 0\},$$

$\omega =$ free ultrafilter on $\mathbb{N}$, $\|T\|_2 = \tau_{\mathcal{R}}(T^*T)^{1/2}$, $\tau_{\mathcal{R}} =$ trace on $\mathcal{R}$, the hyperfinite II_1-factor.

Theorem (Kirchberg '93): Let (M, τ) be a separable finite vN alg with faithful normal tracial state τ. Then M admits a τ-preserving embedding into $\mathcal{R}^\omega$ **iff** $\forall\ \varepsilon > 0$ and every set $u_1, \ldots, u_n$ of unitaries in M, $\exists\ k \geq 1$ and unitaries $v_1, \ldots, v_n$ in $M_k(\mathbb{C})$:

$$\left|\tau(u_j^* u_i) - \mathrm{tr}_k(v_j^* v_i)\right| < \varepsilon, \qquad 1 \leq i, j \leq n.$$

Consider the following sets of $n \times n$ matrices of correlations, $n \geq 2$:

$$\mathcal{G}_{\mathrm{matr}}(n) = \bigcup_{k \geq 1} \Big\{ \big[\mathrm{tr}_k(u_j^* u_i)\big] : u_1, \ldots, u_n \text{ unitaries in } M_k(\mathbb{C}) \Big\},$$

$$\cap|$$

$$\mathcal{G}_{\mathrm{fin}}(n) = \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \ldots, u_n \text{ unitaries in arbitrary finite dim C}^*\text{-alg } (\mathcal{A}, \tau) \Big\},$$

$$\cap|$$

$$\mathcal{G}(n) = \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \ldots, u_n \text{ unitaries in arbitrary finite vN alg } (M, \tau) \Big\}.$$

All sets equal if $n = 2$.

Related sets: $D_{\mathrm{fin}}(n) \subseteq D(n)$ where unitaries are replaced by projections.

Theorem (Kirchberg '93): CEP pos $\iff$ $\mathcal{G}(n) = \mathrm{cl}(\mathcal{G}_{\mathrm{matr}}(n))$, $\forall n \geq 3$.

Theorem (Rørdam-M '19):

1) $\mathcal{G}_{\mathrm{matr}}(n)$ is neither convex, nor closed when $n \geq 3$.
2) $\mathcal{G}_{\mathrm{fin}}(n)$ is convex for all $n \geq 2$, but not closed when $n \geq 11$.
3) $D_{\mathrm{fin}}(n)$ is convex for all $n \geq 2$, but not closed when $n \geq 5$.

To prove 1) and that "$D_{\mathrm{fin}}(n)$ not closed $\Rightarrow \mathcal{G}_{\mathrm{fin}}(2n+1)$ not closed" we used a **trick** (originating in ideas of Regev-Slofstra-Vidick):

Let $p_1, \ldots, p_n$ be projections in a vN alg (M, τ_M) with n.f. tracial state. Define unitaries $u_0, u_1, \ldots u_{2n} \in M$ by $u_0 = 1$ and

$$u_j = 2p_j - 1,\ 1 \le j \le n, \quad u_j = \frac{1}{\sqrt{2}}(u_{j-n} + i \cdot 1),\ n+1 \le j \le 2n.$$

Let (N, τ_N) be some other vN alg with n.f. tracial state. Then $\exists$ unitaries $v_0, v_1, \ldots, v_{2n} \in N$ s.t. $\tau_N(v_j^* v_i) = \tau_M(u_j^* u_i), \forall\, 0 \le i, j \le 2n$, iff $\exists$ projections $q_1, \ldots, q_n \in N$ satisfying

$$\tau_N(q_j q_i) = \tau_M(p_j p_i), \qquad 1 \le i, j \le n.$$

- Recall: If $u \in A$ (unital C*-alg) unitary, then $\frac{1}{\sqrt{2}}(u + i \cdot 1)$ is a unitary **iff** u is a symmetry, i.e., $\frac{1}{2}(u+1)$ is a proj.
- Idea behind the **trick**: the map $u_j \mapsto v_j$, extended linearly between Eucl spaces $(\mathrm{Span}\{u_0, \ldots u_{2n}\}, \langle \cdot, \cdot \rangle_{\tau_M})$, $(\mathrm{Span}\{v_0, \ldots v_{2n}\}, \langle \cdot, \cdot \rangle_{\tau_N})$ is an isometry.

To prove $D_{\rm fin}(n)$ is not closed when $n \geq 5$, we followed ideas of D-P-P '17, and employed a theorem of Kruglyak-Rabanovich-Samoilenko '02, concerning existence of projections on a Hilbert space adding up to a scalar multiple of the identity, to show:

Theorem: Let $n \geq 5$ and $t \in [\frac{1}{2}(1-\sqrt{1-4/n}), \frac{1}{2}(1+\sqrt{1-4/n})]$. Define $A_t^{(n)} = [A_t^{(n)}(i,j)]_{1 \leq i,j \leq n} \in M_n(\mathbb{R})$ by

$$A_t^{(n)}(i,i) = t, \quad A_t^{(n)}(i,j) = \frac{t(nt-1)}{n-1}, i \neq j.$$

If $t \notin \mathbb{Q}$, then $A_t^{(n)} \in \overline{D_{\rm fin}(n)} \setminus D_{\rm fin}(n)$.

► (PSSTW '16): $\mathcal{D}(n)$, $\mathcal{D}_{\rm fin}(n)$ affinely homeo to the sets of *synchronous* quantum correlations $C_{qc}^s(n,2)$, $C_q^s(n,2)$.

Now onto quantum channels:

- (Choi '73): $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ linear is completely positive (cp) iff

$$Tx = \sum_{i=1}^{d} a_i^* x a_i\,, \quad x \in M_n(\mathbb{C})\,,$$

where $a_1\,,\ldots\,,a_d \in M_n(\mathbb{C})$ can be chosen linearly independent.

- The Choi matrix C_T of a linear map $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is

$$C_T = \sum_{i,j=1}^{n} e_{ij} \otimes T(e_{ij}) \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) = M_{n^2}(\mathbb{C}),$$

where $\{e_{ij}\}_{1\leq i,j\leq n}$ are matrix units for $M_n(\mathbb{C})$. Then

$$T(e_{ij}) = \sum_{k,\ell=1}^{n} C_T(i,j;k,\ell)\, e_{k\ell}, \qquad 1 \leq i,j \leq n,$$

where $C_T(i,j;k,\ell) = \langle C_T, e_{ij} \otimes e_{k\ell}\rangle_{\mathrm{Tr}_n\otimes\mathrm{Tr}_n}$ (matrix coefficients).

- (Choi '75): T completely positive **iff** C_T positive matrix.

A cp trace-preserving map $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a quantum channel.

Examples of (unital) quantum channels:

- Automorphisms of $M_n(\mathbb{C})$: $T \in \mathrm{Aut}(M_n(\mathbb{C}))$ iff $\exists u \in \mathcal{U}(M_n(\mathbb{C}))$ s.t.

$$T(x) = u^* x u, \quad x \in M_n(\mathbb{C}).$$

  (Kümmerer '83): Any unital qubit is a convex comb of automorph.

- Completely depolarizing channel S_n, $n \geq 2$

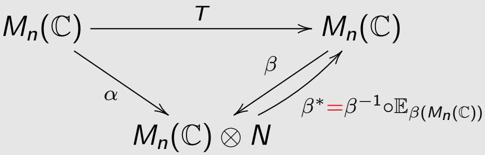
$$S_n(x) = \mathrm{tr}_n(x) 1_n, \quad x \in \mathbb{M}_n(\mathbb{C}).$$

- Schur multipliers associated to (complex) correlation matrices: If $B \in M_n(\mathbb{C})$ is a correlation matrix, then $T_B : M_n(\mathbb{C}) \to M_n(\mathbb{C})$

$$T_B\left([x_{ij}]_{1 \leq i,j \leq n}\right) = [x_{ij} b_{ij}]_{1 \leq i,j \leq n}, \quad [x_{ij}]_{1 \leq i,j \leq n} \in M_n(\mathbb{C}).$$

  is a unital quantum channel.

Definition (Anantharaman-Delaroche '05): A unital quantum channel $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called *factorizable* if $\exists$ vN alg (N, ψ) with n.f. tracial state and unital $*$-homs $\alpha, \beta\colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes N$: $T = \beta^* \circ \alpha$.



▶ α, β are injective (thus embeddings) and trace-preserving. Since unital embeddings of $M_n(\mathbb{C})$ into a vN alg are unitarily equiv, can take

$$\beta(x) = x \otimes 1_N, \quad \alpha(x) = u^* \beta(x) u, \quad x \in M_n(\mathbb{C}),$$

for some $u \in M_n(\mathbb{C}) \otimes N$ unitary. N can be taken II_1-vN alg (even factor).

Theorem (Haagerup-M '11): $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a factorizable quantum channel iff $\exists$ (N, τ_N) finite vN algebra (called ancilla) and a unitary $u \in M_n(\mathbb{C}) \otimes N$: $Tx = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u)$, $x \in M_n(\mathbb{C})$.

Def/Thm (Haagerup-M '11): $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a factorizable quantum channel iff $\exists$ (N, τ_N) finite vN algebra (called ancilla) and a unitary $u \in M_n(\mathbb{C}) \otimes N$: $Tx = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u)$, $x \in M_n(\mathbb{C})$.

▶ (R. Werner): Factorizable channels are obtained by coupling the input system to a maximally mixed ancillary one, executing a unitary rotation on the combined system, and tracing out the ancilla.

▶Automorphisms of $M_n(\mathbb{C})$ (unitarily implem channels) are factorizable.

Let $\mathcal{FM}(n)$ denote all factorizable quantum channels on $M_n(\mathbb{C})$, $n \geq 2$. Then $\mathcal{FM}(n)$ is convex and closed.

Further examples of factorizable channels:

- Convex comb of automorphisms of $M_n(\mathbb{C})$.
- The completely depolarizing channel S_n, as

$$\int_{\mathcal{U}(n)} u^* x u \, d\mu(u) = \mathrm{tr}_n(x) 1_n = S_n(x), \quad x \in M_n(\mathbb{C}).$$

- Schur multipliers associated to **real** correlation matrices (Ricard '08).

Theorem (Haagerup-M '11): For all $n \geq 3$, there exist non-factorizable quantum channels on $M_n(\mathbb{C})$. Each such channel violates the Asymptotic Quantum Birkhoff Conjecture of Smolin-Verstraete-Winter '05.

- Unital quantum channels which are extreme points of CPT or UCP, are non-factorizable. Concrete example: the Holevo-Werner channel W_3^-. With Haagerup and Ruskai, systematic recipe for non-factoriz channels.

- For a factorizable channel, "the" ancilla and its "size" **not** unique. E.g., possible ancillas for S_n are: $\mathbb{C}^{n^2}$, $M_n(\mathbb{C})$, but also (a corner of) $(M_n(\mathbb{C}), \mathrm{tr}_n) * (M_n(\mathbb{C}), \mathrm{tr}_n)$, the reduced free product von Neumann algebra of two copies of $M_n(\mathbb{C})$.

Question: Do we **need** (inf dim) vN alg to describe factorizable channels?

Let $\mathcal{FM}_{\text{fin}}(n)$ = factoriz channels on $M_n(\mathbb{C})$ admitting a finite dim ancilla.

Theorem (Rørdam-M '19): $\mathcal{FM}_{\text{fin}}(n)$ is not closed, whenever $n \geq 11$. Moreover, for each such n, there exist factorizable quantum channels on $M_n(\mathbb{C})$ which do require infinite dimensional (even type II_1) ancilla.

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Proposition (Haagerup-M '11): A Schur multiplier T_B is factorizable iff $B \in \mathcal{G}(n)$ (i.e., $B = [\tau(u_j^* u_i)]$, $u_1, \dots u_n$ unitaries in a fin vN alg (M, τ)). Furthermore,

$$T_B \in \mathcal{FM}_{\mathrm{fin}}(n) \iff B \in \mathcal{G}_{\mathrm{fin}}(n).$$

As the map $B \mapsto T_B$ is an affine homeo, the theorem above follows from non-closure of $\mathcal{G}_{\mathrm{fin}}(n)$, whenever $n \geq 11$.

Thm (Haagerup-M '15) CEP pos **iff** $\overline{\mathcal{FM}_{\text{fin}}(n)} = \mathcal{FM}(n), \forall n \geq 3$.

(Rørdam-M '20): A new view-point on factorizable channels, leading to further connections (and interesting open problems in C*-algebras):

- $\mathcal{FM}(n)$ is *parametrized by* simplex of tracial states $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

More precisely, if $\tau \in T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, let

$$C_\tau(i,j;k,\ell) = n\tau\big(\iota_2(e_{k\ell})^* \, \iota_1(e_{ij})\big), \qquad 1 \le i,j,k,\ell \le n,$$

where $\iota_1, \iota_2 \colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) * M_n(\mathbb{C})$ are the *canonical inclusions*. Then $C_\tau \in M_{n^2}(\mathbb{C})$ is positive, hence it is the Choi matrix of some quantum channel T_τ. Furthermore, turns out that T_τ is factorizable!

In fact, the map $\Phi \colon T(M_n(\mathbb{C}) * M_n(\mathbb{C})) \to \mathcal{FM}(n), \tau \mapsto \Phi(\tau) := T_\tau$ is an affine continuous surjection, satisfying, moreover,

$$\Phi(T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))) = \mathcal{FM}_{\rm fin}(n),$$

where $T_{\rm fin}$ = tracial states that factor through fin. dim. C^*-alg.

The affine cont surj $\Phi\colon T(M_n(\mathbb{C}) * M_n(\mathbb{C})) \to \mathcal{FM}(n), \tau \mapsto T_\tau$, satisfies

- $\Phi(T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))) = \mathcal{FM}_{\rm fin}(n)$,
- $\Phi(\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))}) = \overline{\mathcal{FM}_{\rm fin}(n)}$,

where $T_{\rm fin} =$ tracial states that factor through fin. dim. C^*-alg.

Recall: CEP positive answer $\iff \mathcal{FM}(n) = \overline{\mathcal{FM}_{\rm fin}(n)}$, $\forall n \geq 3$.

Question: What can we say about $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))}$?

- (Exel–Loring '92): $M_n(\mathbb{C}) * M_n(\mathbb{C})$ residually finite dim. (RFD)
- (Blackadar '85): $M_n(\mathbb{C}) * M_n(\mathbb{C})$ semi-projective.

In general, given $A =$ unital C^*-algebra, we have inclusions:

$$T_{\rm fin}(A) \subseteq \overline{T_{\rm fin}(A)} \subseteq T_{\rm qd}(A) \subseteq T_{\rm am}(A) \subseteq T_{\rm hyp}(A) \subseteq T(A),$$

where $T_{\rm qd}(A) =$ quasi-diagonal traces, $T_{\rm am}(A) =$ amenable (=liftable) traces, $T_{\rm hyp}(A) =$ hyperlinear traces (i.e., traces τ st $\pi_\tau(A)'' \hookrightarrow \mathcal{R}^\omega$).

▶ If A is *separable*, then $\overline{T_{\rm fin}(A)}$, $T_{\rm qd}(A)$, $T_{\rm am}(A)$, resp., $T_{\rm hyp}(A)$ contains a faithful trace iff A is RFD, quasi-diagonal, embeds into $\mathcal{R}^\omega$ with ucp lift to $\ell^\infty(\mathcal{R})$, resp., embeds into $\mathcal{R}^\omega$.

- CEP pos answer **iff** $T_{\rm hyp}(A) = T(A)$, for all C^*-alg A.
- It is open whether $T_{\rm qd}(A) = T_{\rm am}(A)$. There are strong positive results!
- (N. Brown '06): $\exists$ exact RFD C^*-alg A s.t. $T_{\rm am}(A) \neq T_{\rm hyp}(A)$.
- A (weakly) semi-projective $\implies$ $\overline{T_{\rm fin}(A)} = T_{\rm qd}(A)$
- (Hadwin–Shulman '17): $\exists$ RFD C^*-alg A s.t. $\overline{T_{\rm fin}(A)} \neq T_{\rm qd}(A)$.

Thm (Rørdam-M '20): $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

Thm (Rørdam-M): $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

Cor: CEP pos **iff** $T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C})) = T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, $\forall\, n \geq 3$.

Further results: Let A be a unital C^*-algebra.

- If $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$, then A is gen by n^2 elem.
- If A is gen by $n-1$ elem, then $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$.

Theorem: Each metrizable Choquet simplex is affinely homeo to a face of $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

Question: Is $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$ the Poulsen simplex?

Groups, C*-tensor norms, Tsirelson's Conjecture, Complexity and **CEP**

For $n, k \geq 2$, let $\Gamma = \mathbb{Z}_n * \mathbb{Z}_n * \cdots * \mathbb{Z}_n$ (k free factors).

Theorem (Kirchberg '93, Fritz/Junge et al '09, Ozawa '12): TFAE:

(i) $C^*(\Gamma) \otimes_{\max} C^*(\Gamma) = C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$, $\forall n, k \geq 2$.

(ii) $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$.

(iii) The Connes Embedding Problem has a positive answer.

(iv) Tsirelson's Conjecture is true: $\mathrm{cl}(\mathcal{C}_{qs}(n, k)) = \mathcal{C}_{qc}(n, k)$, $\forall n, k \geq 2$.

Posted on arXiv, Jan. 13, 2020: $MIP^* = RE$, Ji, Natarajan, Vidick, Wright, Yuen, 165 pp.

Proving that the complexity class MIP* (quantum version of complexity class MIP=languages with a Multiprover Interactive Proof) contains an undecidable language, they conclude that Tsirelson's Conjecture is **false**!

- New version (with corrections) 206 pp., posted on arXiv, Sept. 29, 2020.